

## WEAK QUASISTABILITY AND RAJCHMAN MEASURES

CARLOS KUBRUSLY

*Dedicated to the memory Renée Good (1988–2026) and Alex Pretti (1988–2026)*

ABSTRACT. It is shown that weak quasistability does not imply power boundedness, but coercive power unbounded operators cannot be weakly quasistable. Although a finite measure over the unit disc is a Rajchman measure if and only if the position operator is weakly stable, it is shown that the position operator is weakly quasistable for every finite continuous measure over the unit disc. Corollaries linking Rajchman measures with weak stability and weak quasistability follow the above results.

### 1. INTRODUCTION

Stability of an operator  $T$  (e.g., acting on a normed space) means convergence of the power sequences  $\{T^n\}$  to the null operator. Weak stability holds when this happens in the weak sense. Weak quasistability holds when convergence is weakened to convergence of a subsequence. For recent expositions on weak stability and weak quasistability, see [9] and [14], where these notions are discussed in detail and their importance is explained and exemplified. The present paper investigates the relation of weak stability and weak quasistability with Rajchman measures.

The main results proved here are:

- Weakly quasistable operators are either power bounded or noncoercively power unbounded (Theorem 4.3).
- If  $\{z^k\}$  is an orthonormal basis for  $L^2(\mathbb{T}, \mu)$ , then the measure  $\mu$  is Rajchman, and the converse fails: there are Rajchman measures  $\mu$  for which any pair of distinct elements from  $\{z^k\}$  is not orthogonal in  $L^2(\mathbb{T}, \mu)$  (Theorem 6.1).
- The position operator on  $L^2(\mathbb{T}, \mu)$  is weakly quasistable for every continuous measure  $\mu$  (Theorem 7.3).

Among other conclusions, we highlight the following ones:

- Weakly quasistability does not imply power boundedness (Remark 4.4).
- The position operator on  $L^2(\mathbb{T}, \mu)$  is weakly stable if and only if  $\mu$  is a Rajchman measure, and therefore weak stability is ensured by testing the unit function only (Proposition 5.3).

The paper is organised into 6 more sections. Basic notation and terminology are outlined in Section 2. The definitions of weak stability and weak quasistability are posed and compared in Section 3. Section 4 analyses when a weakly quasistable operator is power bounded. The relation of Rajchman measures with weak stability of the position operator, which is unitary, is explored in Section 5. Section 6 discusses the connection of Rajchman measures  $\mu$  and orthonormal bases for  $L^2(\mathbb{T}, \mu)$ . Section 7 closes the paper by showing that every finite continuous measure over the unit circle is quasi-Rajchman, which implies weak quasistability for the position operator, and also characterises non-Rajchman continuous measures.

---

*Date:* Received September 1, 2025; Accepted January 29, 2026.

*2010 Mathematics Subject Classification.* 47A05, 47B38, 28A25.

*Keywords.* Rajchman measures, unitary operators, weak quasistability, weak stability.

## 2. PRELIMINARIES: NOTATION AND TERMINOLOGY

The following common notation will be adopted throughout the paper. The set of all integers, the set of all positive integers, the real line, and the complex plane are denoted by  $\mathbb{Z}$ ,  $\mathbb{N}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$ , respectively, as usual. Let  $\mathbb{D}$ ,  $\mathbb{D}^-$ , and  $\mathbb{T}$  stand for the open unit disc, the closed unit disc, and the unit circle in  $\mathbb{C}$ , respectively.

A sequence of positive integers is any  $\mathbb{N}$ -valued function on  $\mathbb{N}$ . Regard the set of all positive integers equipped with its natural well-ordering as the self-indexed sequence of all positive integers so that  $\mathbb{N} = \{n\}_{n \geq 1}$ . A subsequence  $\{n_j\} = \{n_j\}_{j \geq 1}$  of the positive integers is a strictly increasing (infinite) sequence of positive integers.

We will identify a subsequence of the positive integers (i.e., a strictly increasing function of  $\mathbb{N}$  into itself) with its range. So a subsequence of the positive integers is identified with an infinite, ordered, strictly increasing subset of  $\mathbb{N}$ . We will use the same notation,  $\{n_j\}$ , for either of them.

A subsequence of the positive integers is *nontrivial* if there is another (equally infinite) subsequence such that they have no common entries. If  $\{n'_j\}$  is a subsequence of a subsequence  $\{n_j\}$  of the positive integers, then  $\{n_j\}$  is a *supersequence* of  $\{n'_j\}$ .

A subsequence  $\{a_{n_j}\}$  of an  $A$ -valued sequence  $\{a_n\}$  (for an arbitrary nonempty set  $A$ ) is the restriction of  $\{a_n\}$  to a subsequence  $\{n_j\}$  of the positive integers.

A subsequence  $\{n_j\}$  of the positive integers is of *bounded increments* (or *bounded gaps*) if  $\sup_j (n_{j+1} - n_j) < \infty$ . A subsequence  $\{a_{n_j}\}$  of  $\{a_n\}$  is *boundedly spaced* if it is indexed by a subsequence  $\{n_j\}$  of bounded increments.

Let  $\mathcal{X}$  be a normed space and let  $\mathcal{X}^*$  be its dual. An  $\mathcal{X}$ -valued sequence  $\{x_n\}$  is weakly convergent if there exists an  $x \in \mathcal{X}$  such that  $\lim_n f(x_n) = f(x)$  for every  $f \in \mathcal{X}^*$ ; equivalently, such that  $\lim_n f(x_n - x) = 0$  for every  $f \in \mathcal{X}^*$ . In this case the sequence  $\{x_n\}$  is said to converge weakly to the vector  $x$ . An alternative and usual notation for weak convergence that will be used here is  $x_n \xrightarrow{w} x$  or  $x = w\text{-}\lim_n x_n$ .

The above definition is standard. Actually, “every normed space has a topology  $\mathcal{T}$  [the weak topology on it] such that a sequence in the space converges weakly to an element of the space if and only if the sequence converges to that element with respect to  $\mathcal{T}$ . For the moment, the statement that a sequence converges weakly to a certain limit should not be taken to imply anything more than is stated [above]” [16, p.116]. We will not deal with weak topology techniques here (although the above definition coincides with convergence in the weak topology in a Hilbert space).

## 3. WEAK QUASISTABILITY: DEFINITION AND COMPARISON

By an operator on a normed space  $\mathcal{X}$  we mean a bounded linear (i.e., a continuous linear) transformation of  $\mathcal{X}$  into itself. Let  $\mathcal{B}[\mathcal{X}]$  stand for the normed algebra of all operators on  $\mathcal{X}$ . The same notation  $\| \cdot \|$  will be used for the norm on  $\mathcal{X}$  and for the induced uniform norm on  $\mathcal{B}[\mathcal{X}]$ . An operator  $T$  is *strongly stable* if  $\lim_n \|T^n x\| = 0$  for every  $x \in \mathcal{X}$  (notation:  $T^n \xrightarrow{s} O$ ), and *uniformly stable* if  $\lim_n \|T^n\| = 0$  (notation:  $T^n \xrightarrow{u} O$ ). An operator  $T$  is *power bounded* if  $\sup_n \|T^n\| < \infty$ ; equivalently, if  $\sup_n \|T^n x\| < \infty$  for every  $x$  in  $\mathcal{X}$  by the Banach–Steinhaus Theorem if  $\mathcal{X}$  is a Banach space. If  $\mathcal{X}$  is a Hilbert space, then it will be denoted by  $\mathcal{H}$ , and  $T^* \in \mathcal{B}[\mathcal{H}]$  will stand for the adjoint of  $T \in \mathcal{B}[\mathcal{H}]$ .

**Definition 3.1.** (a) An operator  $T \in \mathcal{B}[\mathcal{X}]$  is *weakly stable* if, for each  $x \in \mathcal{X}$ ,

$$\lim_n |f(T^n x)| = 0 \text{ for every } f \in \mathcal{X}^*.$$

Notation:  $w\text{-}\lim_n T^n x = 0$ , or  $T^n x \xrightarrow{w} 0$ , for every  $x \in \mathcal{X}$ ; or simply  $T^n \xrightarrow{w} O$ .

(b) Equivalently, an operator  $T$  on a normed space  $\mathcal{X}$  is *weakly stable* if, for each vector  $x \in \mathcal{X}$  and every subsequence  $\{n_j\}$  of the positive integers,

$$\lim_j |f(T^{n_j} x)| = 0 \text{ for every } f \in \mathcal{X}^*.$$

(c) A subsequence  $\{n_j\}$  of the positive integers is a *subsequence of weak stability* of  $T$  for  $x$  if  $\lim_j |f(T^{n_j} x)| = 0$  for every  $f$ . So a weakly stable operator is one for which every subsequence of the positive integers is of weak stability for every vector  $x$ .

An operator is weakly quasistable if the limit in Definition 3.1(a) is weakened to limit inferior.

**Definition 3.2.** (a) An operator  $T \in \mathcal{B}[\mathcal{X}]$  is *weakly quasistable* if, for each  $x \in \mathcal{X}$ ,

$$\liminf_n |f(T^n x)| = 0 \text{ for every } f \in \mathcal{X}^*.$$

Notation:  $w\text{-}\liminf_n T^n x = 0$  for every  $x \in \mathcal{X}$ .

(b) Equivalently, an operator  $T$  on a normed space  $\mathcal{X}$  is *weakly quasistable* if for each vector  $x \in \mathcal{X}$  there exists a subsequence  $\{n_j\} = \{n_j(x)\}$  of the positive integers (that may depend on each  $x$  but does not depend on  $f$ ) for which

$$\lim_j |f(T^{n_j} x)| = 0 \text{ for every } f \in \mathcal{X}^*.$$

(c) So a weakly quasistable operator is an operator  $T$  such that, for every  $x$ , there is at least one subsequence  $\{n_j\}$  (that may depend on  $x$ ) of weak stability of  $T$  for  $x$ .

Weak quasistability plays an important role in weak l-sequential supercyclicity [13, Corollaries 4.3 and 4.4].

It is worth noticing that the notion of quasistability has a plain meaning only in the weak case. In fact, it is easy to see (cf. [14, Propositions 4.1 and 4.2]) that

$$\limsup_n \|T^n\| = 0 \iff \lim_n \|T^n\| = 0$$

(i.e., quasistability always coincides with stability for the uniform case) and

$$\text{if } \sup_n \|T^n\| < \infty, \text{ then } \limsup_n \|T^n x\| = 0 \forall x \in \mathcal{X} \iff \lim_n \|T^n x\| = 0 \forall x \in \mathcal{X}$$

(i.e., for power-bounded operators, quasistability coincides with stability for the strong case). For the weak case the notions are different even for power-bounded operators [14, Proposition 4.3]. However, if a weakly quasistable operator  $T$  is such that for every  $x$  there is a subsequence  $\{n_j\}$  of weak stability that is of bounded increments, then  $T$  is weakly stable [14, Theorem 5.3].

**Definition 3.3.** (a) An operator  $T \in \mathcal{B}[\mathcal{X}]$  is *homogeneously weakly quasistable* if it is weakly quasistable and there exists a subsequence  $\{n_j\}$  of the positive integers such that, for all vectors  $x \in \mathcal{X}$ ,

$$\lim_j |f(T^{n_j} x)| = 0 \text{ for every } f \in \mathcal{X}^*.$$

(b) Equivalently, there is a subsequence of weak stability common to all  $x \in \mathcal{X}$ .

(c) So a homogeneously weakly quasistable is a weakly quasistable operator  $T$  such that at least one subsequence  $\{n_j\}$  of weak stability of  $T$  does not depend on  $x$ .

It is clear that every weakly stable operator is homogeneously weakly quasistable, and every homogeneously weakly quasistable operator is weakly quasistable. An example of a power-bounded operator that is not weakly stable but is homogeneously weakly quasistable will be presented in Remark 7.4.

**Definition 3.4.** (a) An operator  $T$  on a normed space  $\mathcal{X}$  is *weakly unstable* if there exists an  $x_0 \in \mathcal{X}$  such that

$$|f_0(T^n x_0)| \not\rightarrow 0 \text{ as } n \rightarrow \infty \text{ for some } f_0 \in \mathcal{X}^*.$$

Notation:  $T^n \xrightarrow{w} O$ .

(b) Equivalently,  $T \in \mathcal{B}[\mathcal{X}]$  is *weakly unstable* if there exists a vector  $x_0 \in \mathcal{X}$ , a functional  $f_0 \in \mathcal{X}^*$ , and a subsequence  $\{m_j\}$  of the positive integers such that

$$|f_0(T^{m_j} x_0)| \not\rightarrow 0 \text{ as } j \rightarrow \infty.$$

(c) A subsequence  $\{m_j\}$  of the positive integers is a *subsequence of weak instability* of  $T$  for some vector  $x_0$  if it is not a subsequence of weak stability of  $T$  for  $x_0$ . A weakly unstable operator is one for which there exists at least one subsequence of the positive integers that is of weak instability for some vector  $x_0$ .

REMARK 3.5. The particular case of operators acting on a Hilbert space I.

If  $\{T_n\}$  is a  $\mathcal{B}[\mathcal{X}]$ -valued sequence of operators on a complex inner product space  $\mathcal{X}$  with inner product  $\langle \cdot ; \cdot \rangle$ , then the polarisation identity ensures that

$$\begin{aligned} \langle T_n x ; y \rangle &= \frac{1}{4} (\langle T_n(x+y) ; (x+y) \rangle - \langle T_n(x-y) ; (x-y) \rangle \\ &\quad + i \langle T_n(x+iy) ; (x+iy) \rangle - \langle T_n(x-iy) ; (x-iy) \rangle) \end{aligned}$$

for every  $x, y \in \mathcal{X}$ , which implies that the following assertions are equivalent.

- (a)  $\lim_n |\langle T_n x ; y \rangle| = 0$  for every  $x, y \in \mathcal{X}$ .
- (b)  $\lim_n |\langle T_n x ; x \rangle| = 0$  for every  $x \in \mathcal{X}$ .

uch an equivalence still holds in a real Hilbert space if each  $T_n$  is self-adjoint.

REMARK 3.6. The particular case of operators acting on a Hilbert space II.

The Riesz Representation Theorem in a Hilbert space ensures that if  $\mathcal{H}$  is a Hilbert space with inner product  $\langle \cdot ; \cdot \rangle$ , then according to Definitions 3.1 and 3.2, an operator  $T$  on  $\mathcal{H}$  is weakly stable if and only if

- (a) for each  $x \in \mathcal{H}$ ,  $\lim_n |\langle T^n x ; y \rangle| = 0$  for every  $y \in \mathcal{H}$  (i.e., for every  $x, y \in \mathcal{H}$ ).

As we saw in Remark 3.5, if  $\mathcal{H}$  is a complex Hilbert space, then this is equivalent to

- (b)  $\lim_n |\langle T^n x ; x \rangle| = 0$  for every  $x \in \mathcal{H}$ .

Similarly, an operator  $T$  on a Hilbert  $\mathcal{H}$  is weakly quasistable if and only if

- (c) for each  $x \in \mathcal{H}$ ,  $\liminf_n |\langle T^n x ; y \rangle| = 0$  for every  $y \in \mathcal{H}$  (i.e., for every  $x, y \in \mathcal{H}$ ).

That is, for each  $x \in \mathcal{H}$ , there is a subsequence  $\{n_j\}$  of the positive integers such that

- (d)  $\lim_j |\langle T^{n_j} x ; y \rangle| = 0$  for every  $y \in \mathcal{H}$ .

REMARK 3.7. The particular case of operators acting on a Hilbert space III.

(a) Weak stability for an operator  $T$  on a Hilbert space  $\mathcal{H}$  means that the power sequence  $\{T^n\}$  satisfies the equivalent limiting conditions (a) or (b) in Remark 3.6.

Strong stability for an operator  $T$  on a Hilbert space  $\mathcal{H}$  means  $\lim_n \|T^n x\| = 0$  for every  $x \in \mathcal{H}$ , where  $\|\cdot\|$  is the norm generated by the inner product  $\langle \cdot; \cdot \rangle$ , which clearly implies (but is not implied by) weak stability. Differently from the strong stability case, an operator  $T$  and its adjoint  $T^*$  are weakly stable together.

(b) For weak quasistability we cannot expect in general a simplified counterpart with  $y = x$ , as in the weak stability case of Remark 3.6(b). The reason is this: for an arbitrary weakly quasistable operator  $T$ , the subsequence  $\{n_j\}$  appearing in Remark 3.6(d) may depend on  $x$ . Indeed, the polarisation identity displayed in Remark 3.5 only ensures that for each  $x \in \mathcal{H}$  there is a subsequence  $\{n_j(x)\}$  such that

$$\begin{aligned} \langle T^{n_j(x)} x; y \rangle &= \frac{1}{4} (\langle T^{n_j(x)}(x+y); (x+y) \rangle - \langle T^{n_j(x)}(x)(x-y); (x-y) \rangle \\ &\quad + i \langle T^{n_j(x)}(x+iy); (x+iy) \rangle - \langle T^{n_j(x)}(x)(x-iy); (x-iy) \rangle) \end{aligned}$$

for every  $y \in \mathcal{H}$ , and the condition  $\lim_n \langle T^{n_j(x)} x; x \rangle = 0$  for every  $x \in \mathcal{H}$  does not guarantee that the right-hand side goes to zero for every  $y \in \mathcal{H}$ . For homogeneous weak stability, however, a simplified counterpart with  $y = x$ , as in the weak stability case of Remark 3.6(b), holds naturally.

#### 4. WEAK QUASISTABILITY AND POWER BOUNDEDNESS

The following examples set a starting point for a discussion on power boundedness and weak quasistability.

**EXAMPLE 4.1.** A power-bounded, weakly unstable but weakly quasistable operator.

Consider the Foguel operator  $F = \begin{pmatrix} S^* & P \\ O & S \end{pmatrix}$  on the direct sum  $\mathcal{H} \oplus \mathcal{H}$  of a separable infinite-dimensional Hilbert space  $\mathcal{H}$  with itself. Here  $S$  is a unilateral shift of multiplicity one acting on  $\mathcal{H}$  that shifts an orthonormal basis  $\{\mathbf{e}_k\}_{k \geq 0}$  for  $\mathcal{H}$ . The operator  $P: \mathcal{H} \rightarrow \mathcal{H}$  is the orthogonal projection onto the closure of the span of  $\{\mathbf{e}_j: j \in \mathbb{J}\}$ , where  $\mathbb{J}$  is any sparse infinite subset of positive integers with the following property: if  $i, j \in \mathbb{J}$  and  $i < j$ , then  $2i < j$  (e.g.,  $\mathbb{J} = \{j \geq 1: j = 3^k; k \geq 0\}$ , the set of all integral powers of 3, is a sample of a sparse set of positive integers satisfying the above property). The operator  $F$  was the first example of a power-bounded operator that is not similar to a contraction [6, 7]. Moreover, it is also known that  $F$  is not weakly stable. Since, for every  $n \geq 0$ ,

$$F^n = \begin{pmatrix} S^{*n} & P_n \\ O & S^n \end{pmatrix} \quad \text{with} \quad P_{n+1} = \sum_{i=0}^n S^{*n-i} P S^i \quad \text{and} \quad P_0 = O,$$

we get

$$\langle F^{n+1} x; y \rangle = \langle S^{*n+1} x_1, y_1 \rangle + \langle P_{n+1} x_2; y_1 \rangle + \langle S^{n+1} x_2, y_2 \rangle$$

for every  $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$  and every  $y = (y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$ . As  $S$  and  $S^*$  are both weakly stable, it follows that

$$\liminf_n |\langle F^n x; y \rangle| = 0 \quad \iff \quad \liminf_n |\langle P_n x_2; y_1 \rangle| = 0.$$

It was shown in [14, Proposition 4.3] that the power-bounded and weakly unstable operator  $F$  is weakly quasistable by exhibiting a subsequence  $\{n_j\}$  of the positive integers such that each  $P_{n_j} x_2$  is orthogonal to  $y_1$  for every  $x_2, y_1 \in \mathcal{H}$ , and so  $\{\langle P_{n_j} x_2; y_1 \rangle\}$  is a null subsequence.

**REMARK 4.2.** Spectral properties and power boundedness in weak quasistability.

(a) As in the weak stability case, weak quasistability is also preserved under the adjoint operation. An operator  $T$  and its adjoint  $T^*$ , both acting on a Hilbert space

$\mathcal{H}$ , are weakly quasistable together because

$$\liminf_n |\langle T^n x; y \rangle| = \liminf_n |\langle T^{*n} y; x \rangle| \text{ for every } x, y \in \mathcal{H}.$$

(b) Let  $\{\sigma_P(T), \sigma_R(T), \sigma_C(T)\}$  be the classical partition of the spectrum  $\sigma(T)$  of  $T$  consisting of point spectrum, residual spectrum, and continuous spectrum, respectively. Since  $T$  and  $T^*$  are weakly quasistable together and  $\sigma_R(T) = \sigma_P(T^*)^* \setminus \sigma_P(T)$ , with  $\Lambda^*$  denoting the set of all complex conjugates of an arbitrary set  $\Lambda \subset \mathbb{C}$ , it follows that  $\liminf_n |\langle T^n x; y \rangle| = 0$  for every  $x, y \in \mathcal{H}$  implies  $(\sigma_R(T) \cup \sigma_P(T)) \cap \mathbb{T} = \emptyset$ . Thus, as in the case of weak stability,

$$T \text{ is weakly quasistable} \implies \sigma(T) \cap \mathbb{T} \subseteq \sigma_C(T).$$

The Gelfand–Beurling formula for the spectral radius shows at once that a power-bounded operator is such that  $\sigma(T) \subseteq \mathbb{D}^-$ . Therefore,

$$T \text{ is weakly quasistable and power bounded} \implies \sigma(T) \cap \mathbb{T} \subseteq \sigma_C(T) \subseteq \mathbb{D}^-.$$

If an operator  $T$  on a Banach space is power unbounded (i.e., not power bounded), then there is an  $x_0 \in \mathcal{X}$  for which  $\sup_n \|T^n x_0\| = \infty$  by the Banach–Steinhaus Theorem. Actually, the Banach–Steinhaus Theorem implies that there is a dense  $G_\delta$ -set of such points. Being power unbounded, however, is not enough to ensure that there exists a vector  $x_0$  for which  $\lim_n \|T^n x_0\| = \infty$ . (A detailed discussion along this line can be found in, e.g., [3, Sections III.2 and III.4, pp.48,66].)

For lack of a better name, we call a power unbounded operator  $T$  *iterated coercive*, or simply *coercive* for short, if there is a vector  $x_0$  in the normed space  $\mathcal{X}$  such that  $\lim_n \|T^n x_0\| = \infty$ ; otherwise it will be called *noncoercive*. In any case, if  $T$  is power unbounded, then for every vector  $x_0$  such that  $\sup_n \|T^n x_0\| = \infty$ , there exists a subsequence  $\{m_j\}$  of the positive integers such that  $\lim_j \|T^{m_j} x_0\| = \infty$ , which will be called a *subsequence of coercivity* of  $T$  for the vector  $x_0$ .

**Theorem 4.3.** *A weakly quasistable operator  $T$  on a Banach space  $\mathcal{X}$  is either*

(a) *power bounded*

*or*

(b<sub>1</sub>) *noncoercive power unbounded.*

*Moreover, if it is a noncoercive power unbounded operator, then*

(b<sub>2</sub>) *for any vector  $x_0 \in \mathcal{X}$  such that  $\sup_n \|T^n x_0\| = \infty$ , every pair of subsequences, one of coercivity for  $x_0$  and the other of weak stability for  $x_0$ , has a finite set of common entries.*

*Proof.* First recall that if  $\{x_n\}$  is a weakly convergent sequence with entries in a normed space  $\mathcal{X}$ , then it is bounded; that is,

$$x_n \xrightarrow{w} x \text{ for some } x \in \mathcal{X} \implies \sup_n \|x_n\| < \infty. \quad (§)$$

Now let  $\mathcal{X}$  be a Banach space and let  $T$  be an operator on  $\mathcal{X}$ .

(a) The above implication and an application of the Banach–Steinhaus Theorem ensure the well-known relation:

$$\text{weak stability implies power boundedness.}$$

Furthermore, there also exist power-bounded weakly quasistable operators that are not weakly stable (see, e.g., Example 4.1). This settles the power-bounded case.

(b) Suppose  $T$  is not power bounded. Then there exists a vector  $x_0$  in  $\mathcal{X}$  for which  $\sup_n \|T^n x_0\| = \infty$ . So there is a subsequence  $\{m_j\}$  of the positive integers such that

$$\lim_j \|T^{m_j} x_0\| = \infty. \quad (\dagger)$$

If  $T$  is weakly quasistable, then there also exists a subsequence  $\{n_j\}$  such that

$$T^{n_j} x_0 \xrightarrow{w} 0$$

and hence, by (§),

$$\sup_j \|T^{n_j} x_0\| < \infty. \quad (\ddagger)$$

(b<sub>1</sub>) If  $T$  is a coercive power unbounded operator, then we can take a vector  $x_0$  for which the subsequence  $\{m_j\}$  in (†) may be replaced by the whole sequence  $\{n\}$  of all positive integers to get

$$\lim_n \|T^n x_0\| = \infty,$$

and so there is no subsequence of the positive integers that does not satisfy the above limit. Hence the inequality in (‡) leads to a contradiction. Consequently, if  $T$  is weakly quasistable and power unbounded, then it is noncoercive.

(b<sub>2</sub>) Next suppose  $T$  is a noncoercive power unbounded. If the subsequences  $\{m_j\}$  and  $\{n_j\}$  in (†) and (‡) have an infinite set of common pairwise distinct entries, then there exists a subsequence of the positive integers made of elements from  $\{m_j\} \cap \{n_j\}$ , which is a subsequence of both  $\{m_j\}$  and  $\{n_j\}$ . This subsequence of both  $\{m_j\}$  and  $\{n_j\}$  leads to another contradiction according to (†) and (‡).  $\square$

REMARK 4.4. Power unbounded weakly quasistable operators.

As is also widely known, if  $T$  is a Banach-space operator, then

$$\begin{aligned} r(T) < 1 &\iff T^n \xrightarrow{u} O \implies T^n \xrightarrow{s} O \implies T^n \xrightarrow{w} O \\ &\implies \sup_n \|T^n\| < \infty \implies r(T) \leq 1, \end{aligned}$$

where  $r(T)$  stands for the spectral radius of  $T$ . In particular, strong stability implies weak stability, which in turn implies power boundedness:

$$\begin{aligned} \lim_n \|T^n x\| = 0 \quad \forall x \in \mathcal{X} &\implies \forall x \in \mathcal{X}, \lim_n |f(T^n x)| = 0 \quad \forall f \in \mathcal{X}^* \\ &\implies \sup_n \|T^n x\| < \infty \quad \forall x \in \mathcal{X}. \end{aligned}$$

Also, if  $T$  is a power-bounded operator on a normed space  $\mathcal{X}$ , then strong quasistability coincides with strong stability [14, Proposition 4.2], that is,

$$\sup_n \|T^n\| < \infty \implies \left\{ \lim_n \|T^n x\| = 0 \quad \forall x \in \mathcal{X} \iff \liminf_n \|T^n x\| = 0 \quad \forall x \in \mathcal{X} \right\},$$

differently from the weak case, where there are power-bounded weakly quasistable operators that are not weakly stable, as we saw in Example 4.1. However, even for the strong case, the equivalence collapses for a power unbounded operator:

$$\liminf_n \|T^n x\| = 0 \quad \forall x \in \mathcal{X} \not\implies \sup_n \|T^n\| < \infty \quad \& \quad \sup_n \|T^n\| = \infty \implies T^n \xrightarrow{s} O.$$

Indeed, there exists a diagonal operator  $T$  on the Hilbert space  $\ell_+^2$  with

$$\|T^n\| = (\log n)^{\frac{1}{2}} \quad \forall n \in \mathbb{N} \quad \text{and} \quad \inf_n \|T^n x\| = 0 \quad \forall x \in \ell_+^2$$

[3, Section III.4.A, p.66]. Since  $T$  is power unbounded, it is weakly unstable and so strongly unstable. Since, for every  $x \in \ell_+^2$ ,  $\inf_n \|T^n x\| = 0$  implies  $\liminf_n \|T^n x\| = 0$  (because  $\|T^{n+1}x\| \leq \|T\| \|T^n x\|$ ), we get  $\liminf_n |\langle T^n x; y \rangle| = 0$  for every  $x, y \in \ell_+^2$  (by the Schwartz inequality). So the operator  $T$  is power unbounded and strongly

quasistable and so weakly quasistable. Summing up: (i) for power unbounded operators, strong quasistability does not imply strong stability, and, moreover, (ii)

weak quasistability does not imply power boundedness.

### 5. RAJCHMAN MEASURES AND WEAK STABILITY

The  $\sigma$ -algebra of all Borel subsets of the real line  $\mathbb{R}$  will be denoted by  $\mathfrak{R}$ . With  $\wp$  standing for power set,  $\mathfrak{R}_{|0,1)} = \mathfrak{R} \cap \wp([0, 1))$  denotes the  $\sigma$ -algebra of all Borel subsets of the interval  $[0, 1)$ , and  $\mathfrak{R}_{\mathbb{T}}$  stands for the  $\sigma$ -algebra of all Borel subsets of the unit circle  $\mathbb{T}$ . All measures in this paper are positive. The terms “absolutely continuous”, “continuous”, “singular”, and “discrete” for a given measure are with respect to normalised Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$  (or on  $\mathfrak{R}_{|0,1)}$ , unless otherwise stated.

For each integer  $k \in \mathbb{Z}$ , consider the function  $\mathbf{e}_k: \mathbb{T} \rightarrow \mathbb{T}$  given by  $\mathbf{e}_k(z) = z^k$  for every  $z \in \mathbb{T}$ . For simplicity, write  $z^k$  instead of  $\mathbf{e}_k$ , as usual, and let the countable collection of (trigonometric) functions  $\{\mathbf{e}_k\}_{k \in \mathbb{Z}}$  be written simply as  $\{z^k\}$ .

A finite measure  $\mu: \mathfrak{R}_{\mathbb{T}} \rightarrow \mathbb{R}$  is a *Rajchman measure* if

$$\int_{\mathbb{T}} z^k d\mu \rightarrow 0 \text{ as } |k| \rightarrow \infty.$$

The term comes from the pioneering works of Rajchman on this class of measures (e.g., [19, §5]). Since Borel measures over a compact metric space are regular (see, e.g., [4, Corollary 10.6]), Rajchman measures are regular because they are finite (and so Borel) acting over the compact set  $\mathbb{T}$ . Take the measure  $\eta: \mathfrak{R}_{|0,1)} \rightarrow \mathbb{R}$  induced by the measure  $\mu: \mathfrak{R}_{\mathbb{T}} \rightarrow \mathbb{R}$  via a function  $\gamma: [0, 1) \rightarrow \mathbb{T}$ , defined by  $\eta(A) = \mu(\gamma(A))$  for every  $A \in \mathfrak{R}_{|0,1)}$ . Here the function  $\gamma$  is given by  $\gamma(\alpha) = e^{2\pi i \alpha}$  for every  $\alpha \in [0, 1)$ , which is measurable, invertible, with a measurable inverse. The above integral sets the Fourier transform  $\tilde{\eta}$  of  $\eta$ , namely,

$$\tilde{\eta}(k) = \int_{\mathbb{T}} z^k d\mu(z) = \int_{|0,1)} e^{2\pi i k \alpha} d\eta(\alpha) \text{ for every } k \in \mathbb{Z}.$$

For a survey on Rajchman measures, see [15] (also [10, Chapter IX]). The basic properties required here are listed below (see, e.g., [15, p.364 and Theorem 3.4]).

*Every absolutely continuous finite measure is Rajchman*

(in particular, the normalised Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$  is Rajchman).

*Every Rajchman measure is continuous.*

*There exist singular Rajchman measures*

(and so every singular Rajchman measure is singular-continuous).

That every absolutely continuous finite measure is Rajchman is readily verified by the Riemann–Lebesgue Lemma and the Radon–Nikodym Theorem. That every Rajchman measure is continuous is a direct consequence of Neder’s answer [18, Section 1] to a question posed by Riesz [20, p.315].

The Cantor–Lebesgue measure  $\eta_{|0,1)}$  on  $\mathfrak{R}_{|0,1)}$  is a classical example of a singular-continuous measure on  $\mathfrak{R}_{|0,1)}$ . This is the Borel-Stieltjes measure on  $\mathfrak{R}_{|0,1)}$  generated by the Cantor function associated with the Cantor set in  $[0, 1]$ . The Cantor–Lebesgue measure  $\mu_{\mathbb{T}}$  on  $\mathfrak{R}_{\mathbb{T}}$  is obtained from  $\eta_{|0,1)}$  as  $\mu_{\mathbb{T}}(E) = \mu_{|0,1)}(\gamma^{-1}(E))$  for  $E \in \mathfrak{R}_{\mathbb{T}}$ , where the transformation  $\gamma: [0, 1) \rightarrow \mathbb{T}$  was defined above. Thus  $\mu_{\mathbb{T}}$  is a singular-continuous measure on  $\mathfrak{R}_{\mathbb{T}}$ . This, however, is not a Rajchman measure. Indeed, it can be verified that  $\int_{\mathbb{T}} z^k d\mu_{\mathbb{T}} = \int_{\mathbb{T}} z^{3k} d\mu_{\mathbb{T}}$  (i.e.,  $\tilde{\eta}_{\mathbb{T}}(k) = \tilde{\eta}_{\mathbb{T}}(3k)$ ) for all  $k \in \mathbb{Z}$ . The first

example of a singular Rajchman measure is due to Menchoff [17, Lemma p.433 and Theorem p.435], which is a modification of the Cantor–Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$ .

Rajchman measures can be equivalently defined as

$$\int_{\mathbb{T}} z^n d\mu \rightarrow 0 \text{ as } n \rightarrow \infty$$

with  $n$  running over the positive integers. Indeed,  $\overline{\int_{\mathbb{T}} z^n d\mu} = \int_{\mathbb{T}} \overline{z^n} d\mu = \int_{\mathbb{T}} z^{-n} d\mu$  for every  $n \in \mathbb{N}$  whenever  $z \in \mathbb{T}$ , so that  $\int_{\mathbb{T}} z^n d\mu \rightarrow 0$  as  $n \rightarrow \infty$  implies  $\int_{\mathbb{T}} z^k d\mu \rightarrow 0$  as  $|k| \rightarrow \infty$ , and the reverse implication is trivial.

Let  $\mu$  be a finite measure on  $\mathfrak{R}_{\mathbb{T}}$ . Consider the Hilbert space  $L^2(\mathbb{T}, \mu)$  of all complex functions on  $\mathbb{T}$  that are square-integrable with respect to  $\mu$ . Let  $\varphi: \mathbb{T} \rightarrow \mathbb{T}$  denote the identity function (i.e.,  $\varphi(z) = z$   $\mu$ -a.e. for  $z \in \mathbb{T}$ ). Take the multiplication operator  $U_{\varphi, \mu}$  on  $L^2(\mathbb{T}, \mu)$  induced by the identity function  $\varphi$ , also referred to as the *position operator* [8, p.89], which is defined by

$$(U_{\varphi, \mu} \psi)(z) = \varphi(z)\psi(z) = z\psi(z) \quad \mu\text{-a.e. for } z \in \mathbb{T} \quad \text{for every } \psi \in L^2(\mathbb{T}, \mu).$$

So  $(U_{\varphi, \mu} \psi)(z) = z\psi(z)$  and  $(U_{\varphi, \mu}^* \psi)(z) = \overline{z}\psi(z) = z^{-1}\psi(z)$  for every  $\psi \in L^2(\mathbb{T}, \mu)$  and every  $z \in \mathbb{T}$ . Thus  $U_{\varphi, \mu}^* U_{\varphi, \mu} = U_{\varphi, \mu} U_{\varphi, \mu}^* = I$ , and  $U_{\varphi, \mu}$  is unitary on  $L^2(\mathbb{T}, \mu)$ . From now on,  $U_{\varphi, \mu}$  will always denote such a multiplication operator (i.e., the position operator) on  $L^2(\mathbb{T}, \mu)$  for some measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$ . With  $\langle \cdot ; \cdot \rangle$  standing for the inner product in  $L^2(\mathbb{T}, \mu)$ , we get for every  $\psi, \phi \in L^2(\mathbb{T}, \mu)$  and every  $k \in \mathbb{Z}$ ,

$$\langle U_{\varphi, \mu}^k \psi ; \phi \rangle = \int_{\mathbb{T}} z^k \psi \overline{\phi} d\mu \quad \text{so that} \quad \langle U_{\varphi, \mu}^k 1 ; 1 \rangle = \int_{\mathbb{T}} z^k d\mu,$$

as  $U_{\varphi, \mu}$  is invertible. For the second identity, also because the unit function  $1: \mathbb{T} \rightarrow \mathbb{T}$  (i.e.,  $1(z) = 1$   $\mu$ -a.e. for  $z \in \mathbb{T}$ ) lies in  $L^2(\mathbb{T}, \mu)$  since the measure  $\mu$  is finite.

Before proving the next proposition, we need the following two well-known auxiliary results, which will be often required in the sequel.

**REMARK 5.1.** A standard application of the Stone–Weierstrass Theorem.

Let  $P(\mathbb{T})$  denote the set of all polynomials  $q(\cdot, \cdot): \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{C}$  in  $z$  and  $\overline{z}$  (of the form  $\sum_{k, \ell=0}^N \alpha_{k, \ell} z^k \overline{z}^{\ell}$  with  $z^k, \overline{z}^{\ell} \in \mathbb{T}$  and  $\alpha_{k, \ell} \in \mathbb{C}$ .) This is a classical density result:

$P(\mathbb{T})$  is dense in  $(L^p(\mathbb{T}, \mu), \|\cdot\|_p)$  for every finite measure  $\mu$  in  $\mathfrak{R}_{\mathbb{T}}$ ,  
and therefore  $(L^p(\mathbb{T}, \mu), \|\cdot\|_p)$  is separable, for every  $p \geq 1$ .

Indeed, since  $\mathbb{T}$  is *compact* in  $\mathbb{C}$ , the Stone–Weierstrass Theorem ensures that  $P(\mathbb{T})$  is dense in the linear space  $C(\mathbb{T})$  of all complex-valued continuous functions on  $\mathbb{T}$  equipped with the sup-norm. Thus, since the measure  $\mu$  is *finite*,  $P(\mathbb{T})$  is dense in  $C(\mathbb{T})$  equipped with the norm- $p$  for any  $p \geq 1$ . Also, since on compact metric spaces Borel measures (in particular, finite measures) are regular, the set  $C(\mathbb{T})$  is dense in  $(L^p(\mathbb{T}, \mu), \|\cdot\|_p)$  for every  $p \geq 1$  (see, e.g., [2, Theorem 29.14]). By transitivity,  $P(\mathbb{T})$  is dense in  $(L^p(\mathbb{T}, \mu), \|\cdot\|_p)$ . Therefore, as  $P(\mathbb{T})$  is a linear span of a countable set,  $(L^p(\mathbb{T}, \mu), \|\cdot\|_p)$  is separable.

**REMARK 5.2.** A basic change of variables for complex measurable functions.

Let  $\mu$  be a measure on  $\mathfrak{R}_{\mathbb{T}}$ , and let  $g$  be a nonnegative  $\mathfrak{R}_{\mathbb{T}}$ -measurable function on  $\mathbb{T}$ . Take the measure  $\eta_g$  on  $\mathfrak{R}_{\mathbb{T}}$  generated by  $\mu$  via  $g$  defined by  $\eta_g(E) = \int_E g d\mu$  for every

$E$  in  $\mathfrak{R}_{\mathbb{T}}$ . Recall the elementary identity  $\int_E hg d\mu = \int_E h d\eta_g$  for every  $E$  in  $\mathfrak{R}_{\mathbb{T}}$  and every nonnegative  $\mathfrak{R}_{\mathbb{T}}$ -measurable function  $h$  on  $\mathbb{T}$ . Also, for an arbitrary  $\mathbb{C}$ -valued  $\mathfrak{R}_{\mathbb{T}}$ -measurable function  $\zeta$  on  $\mathbb{T}$ , consider its natural decomposition into nonnegative parts given by  $\zeta = h_1 - h_2 + i(h_3 - h_4)$ , where  $h_1, h_2, h_3,$  and  $h_4$  are nonnegative  $\mathfrak{R}_{\mathbb{T}}$ -integrable functions on  $\mathbb{T}$ . So we get the identity for complex functions:

$$\int_E \zeta g d\mu = \int_E \zeta d\eta_g \quad \text{for every } E \in \mathfrak{R}_{\mathbb{T}}.$$

The equivalence between (a) and (e) in Proposition 5.3 below was mentioned in [1, p.1383] and used in [1, Theorem 5], [12, Proposition 3.3], and [9, Eq. (5.3)].

**Proposition 5.3.** *If  $\mu$  is a finite measure on  $\mathfrak{R}_{\mathbb{T}}$  and  $U_{\varphi, \mu}$  is the position operator on  $L^2(\mathbb{T}, \mu)$ , then the following assertions are pairwise equivalent.*

- (a)  $\mu$  is a Rajchman measure.
- (b)  $\int_{\mathbb{T}} z^k g d\mu \rightarrow 0$  as  $|k| \rightarrow \infty$  for every  $g \in L^p(\mathbb{T}, \mu)$ , for every  $p \geq 1$ .
- (c)  $\int_{\mathbb{T}} z^k g d\mu \rightarrow 0$  as  $|k| \rightarrow \infty$  for every  $g \in L^p(\mathbb{T}, \mu)$ , for some  $p \geq 1$ .
- (d)  $\int_{\mathbb{T}} z^k g d\mu \rightarrow 0$  as  $|k| \rightarrow \infty$  for an arbitrary positive  $g \in L^\infty(\mathbb{T}, \mu)$ .
- (e)  $U_{\varphi, \mu}$  is weakly stable.
- (f)  $\langle U_{\varphi, \mu}^n \psi; \psi \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for an arbitrary  $\psi \in L^\infty(\mathbb{T}, \mu)$  with  $0 < |\psi|$ .
- (g)  $\langle U_{\varphi, \mu}^n 1; 1 \rangle \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* (a)  $\implies$  (b). Let  $q = \sum_{k, \ell=0}^N \alpha_{k, \ell} z^k z^{-\ell}$  be an arbitrary polynomial in  $P(\mathbb{T})$  so that  $\int_{\mathbb{T}} z^n q d\mu = \sum_{k, \ell=0}^N \alpha_{k, \ell} \int_{\mathbb{T}} z^n z^k z^{-\ell} d\mu$ . If  $\mu$  is a Rajchman measure, then

$$\int_{\mathbb{T}} z^{n+k-\ell} d\mu \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ for each } k, \ell \in \mathbb{Z}.$$

Thus  $\int_{\mathbb{T}} z^n q d\mu \rightarrow 0$  for every  $q \in P(\mathbb{T})$ . As we saw in Remark 5.1,  $P(\mathbb{T})$  is dense in  $(L^p(\mathbb{T}, \mu), \|\cdot\|_p)$  for every  $p \geq 1$ . Hence  $\lim_n \int_{\mathbb{T}} z^n g d\mu = 0$ , and since  $\int_{\mathbb{T}} z^k g d\mu = \int_{\mathbb{T}} z^{-k} \bar{g} d\mu$ , we get  $\int_{\mathbb{T}} z^k g d\mu \rightarrow 0$  as  $|k| \rightarrow \infty$ , for every  $g \in L^p(\mathbb{T}, \mu)$ , for every  $p \geq 1$ .

(b)  $\implies$  (c) trivially and (c)  $\implies$  (d) since  $L^\infty(\mathbb{T}, \mu) \subset L^p(\mathbb{T}, \mu)$  for  $p \geq 1$  as  $\mu$  is finite.

(d)  $\implies$  (a). Suppose (d) holds. Take an arbitrary positive  $g \in L^\infty(\mathbb{T}, \mu)$ , and let  $g^{-1} = \frac{1}{g}$  denote its reciprocal, which is again a positive function in  $L^\infty(\mathbb{T}, \mu)$ , so that both  $g$  and  $g^{-1}$  lie in  $L^p(\mathbb{T}, \mu)$  for every  $p \geq 1$ , once  $\mu$  is finite. Since  $g$  is positive and measurable, take the measure  $\eta_g$  on  $\mathfrak{R}_{\mathbb{T}}$  defined by  $\eta_g(E) = \int_E g d\mu$  for every  $E$  in  $\mathfrak{R}_{\mathbb{T}}$  so that  $\eta_g$  is absolutely continuous with respect to  $\mu$  (i.e.,  $\eta_g \ll \mu$ ) and finite as  $g$  lies in  $L^1(\mathbb{T}, \mu)$ . Also, by Remark 5.2 with  $\zeta = z^k$ , we get  $\int_E z^k d\eta_g = \int_E z^k g d\mu$  for every  $E \in \mathfrak{R}_{\mathbb{T}}$  and  $k \in \mathbb{Z}$ . Hence, with  $E = \mathbb{T}$ , (d) implies  $\int_{\mathbb{T}} z^k d\eta_g \rightarrow 0$  as  $|k| \rightarrow \infty$ , and so  $\eta_g$  is Rajchman. Applying Remark 5.2 again, now with  $\zeta = z^k g^{-1}$ , it follows that

$$\int_E z^k g^{-1} d\eta_g = \int_E z^k g^{-1} g d\mu = \int_E z^k d\mu \quad (\star)$$

for every  $E$  in  $\mathfrak{R}_{\mathbb{T}}$  and every  $k$  in  $\mathbb{Z}$ . There are two ways to conclude the proof.

Conclusion 1. As  $\eta_g$  is Rajchman and  $g^{-1} \in L^1(\mathbb{T}, \mu)$ ,  $\int_{\mathbb{T}} z^k g^{-1} d\eta_g \rightarrow 0$  as  $|k| \rightarrow \infty$  because (a)  $\implies$  (c). So  $\mu$  is Rajchman by  $(\star)$  with  $E = \mathbb{T}$ . Thus (a) holds true.

Conclusion 2. For  $k = 0$ ,  $(\star)$  says that  $\mu(E) = \int_E d\mu = \int_E g^{-1} d\eta_g$  for every  $E \in \mathfrak{R}_{\mathbb{T}}$ , and so  $\mu$  is absolutely continuous with respect to  $\eta_g$  (i.e.,  $\mu \ll \eta_g$ ). Since  $\eta_g$  is Rajchman, it follows that  $\mu$  must be Rajchman. (This is a result due to Milicer-Grużewska (cf. [15, p.365 (2.1)]). Thus, again, (a) holds true.

(a)  $\implies$  (e). Take an arbitrary  $\psi \in L^2(\mathbb{T}, \mu)$ . Thus  $|\psi|^2 \in L^1(\mathbb{T}, \mu)$ . If  $\mu$  is Rajchman, then  $\lim_n \int_{\mathbb{T}} z^n |\psi|^2 d\mu = 0$  since (a)  $\implies$  (b). Equivalently,  $\lim_n \int_{\mathbb{T}} U_{\varphi, \mu}^n |\psi|^2 d\mu = 0$ , which means  $\lim_n \langle U_{\varphi, \mu}^n \psi; \psi \rangle = 0$ . Since this holds for every  $\psi \in L^2(\mathbb{T}, \mu)$ , and since  $L^2(\mathbb{T}, \mu)$  is a complex Hilbert space, it follows that  $\lim_n \langle U_{\varphi, \mu}^n \psi; \phi \rangle = 0$  for every  $\psi, \phi \in L^2(\mathbb{T}, \mu)$  (cf. Remark 3.6(a,b)); that is,  $U_{\varphi, \mu}$  is weakly stable.

(e)  $\implies$  (f) since  $L^\infty(\mathbb{T}, \mu) \subset L^2(\mathbb{T}, \mu)$  because  $\mu$  is finite.

(f)  $\implies$  (a) because  $\langle U_{\varphi, \mu}^n \psi; \psi \rangle = \int_{\mathbb{T}} z^n |\psi|^2 d\mu$  and (d)  $\implies$  (a).

(g)  $\iff$  (a) as  $\int_{\mathbb{T}} z^n d\mu = \langle U_{\varphi, \mu}^n 1; 1 \rangle$  and  $1 \in L^2(\mathbb{T}, \mu)$  because  $\mu$  is finite.  $\square$

## 6. RAJCHMAN MEASURES AND TRIGONOMETRIC BASIS

The celebrated Riemann–Lebesgue Lemma says that

$$\int_{\mathbb{T}} z^k \psi(z) d\lambda \rightarrow 0 \text{ as } |k| \rightarrow \infty \text{ for every } \psi \in L^2(\mathbb{T}, \lambda)$$

if  $\lambda$  is the normalised Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$ .

There are different proofs of the Riemann–Lebesgue Lemma. One of them that uses a bit of operator theory, whose argument is required in the proof of Theorem 6.1 below, is this. As observed in Remark 5.1, since  $\mathbb{T}$  is compact and  $\lambda$  is finite,  $L^2(\mathbb{T}, \lambda)$  is separable. Take the position operator  $U_{\varphi, \lambda}$  on  $L^2(\mathbb{T}, \lambda)$  so that  $\int_{\mathbb{T}} z^k \psi(z) d\lambda(z) = \langle U_{\varphi, \lambda}^k \psi; 1 \rangle$ , where  $\psi$  and the unit function 1 are in  $L^2(\mathbb{T}, \lambda)$ . Recall that (i)

$\lambda$  is the normalised Lebesgue measure  $\implies \{z^k\}$  is an orthonormal basis for  $L^2(\mathbb{T}, \lambda)$

(this is a standard result; see, e.g., [8, p.18]), that (ii) if an operator shifts an orthonormal basis that runs over  $\mathbb{Z}$ , then it is a bilateral shift of multiplicity one, and also that (iii) every bilateral shift on a Hilbert space is weakly stable. Thus, since  $U_{\varphi, \mu}$  shifts  $\{z^k\}_{k \in \mathbb{Z}}$  independently of the measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$ ,

$$\{z^k\}_{k \in \mathbb{Z}} \text{ is an orthonormal basis for } L^2(\mathbb{T}, \mu) \implies U_{\varphi, \mu} \text{ is weakly stable.}$$

Therefore,  $\langle U_{\varphi, \lambda}^n \psi; \phi \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for every  $\psi, \phi \in L^2(\mathbb{T}, \lambda)$ , and so  $\int_{\mathbb{T}} z^n d\lambda = \langle U_{\varphi, \lambda}^n 1; 1 \rangle \rightarrow 0$  as  $|n| \rightarrow \infty$ . This paves the way for the next result.

**Theorem 6.1.** *Let  $\mu$  be a measure on  $\mathfrak{R}_{\mathbb{T}}$ .*

- (a) *If  $\{z^k\}$  is an orthonormal basis for  $L^2(\mathbb{T}, \mu)$ , then  $\mu$  is a Rajchman probability measure.*

*The converse of (a) fails.*

- (b) *If a probability measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$  is singular-continuous and concentrated on the first quadrant of  $\mathbb{T}$ , then  $\{z^k\}$  is a set of pairwise nonorthogonal unit vectors in  $L^2(\mathbb{T}, \mu)$ .*

*This implies the next result.*

- (c) *There exist Rajchman probability measures  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$  for which the set  $\{z^k\}$  of unit vectors is not orthogonal in  $L^2(\mathbb{T}, \mu)$ .*

*Proof.* (a<sub>1</sub>) With  $\mathbf{e}_k(z) = z^k \in \mathbb{T}$  for every  $z \in \mathbb{T}$  and  $\{\mathbf{e}_k\}$  being an orthonormal set, we get  $1 = \|\mathbf{e}_k\|^2 = \int_{\mathbb{T}} |\mathbf{e}_k(z)|^2 d\mu(z) = \int_{\mathbb{T}} |z^k|^2 d\mu(z) = \int_{\mathbb{T}} d\mu = \mu(\mathbb{T})$  for every  $k$  in  $\mathbb{Z}$ , so that  $\mu$  is a probability measure.

(a<sub>2</sub>) Using the preceding argument, we can verify that  $\int_{\mathbb{T}} z^n d\mu \rightarrow 0$  as  $n \rightarrow \infty$ . In fact, let  $U_{\varphi, \mu}$  be the position operator on  $L^2(\mathbb{T}, \mu)$ . If  $\{z^k\}$  is an orthonormal basis for  $L^2(\mathbb{T}, \mu)$ , then  $U_{\varphi, \mu}$  is a bilateral shift (because it shifts this orthonormal basis; i.e.,  $U_{\varphi, \mu} \mathbf{e}_k = \mathbf{e}_{k+1}$ ). Thus, like every bilateral shift, it is weakly stable, so that  $\langle U_{\varphi, \mu}^n \mathbf{1}; \mathbf{1} \rangle = \int_{\mathbb{T}} z^n d\mu \rightarrow 0$  as  $n \rightarrow \infty$ . This implies  $\int_{\mathbb{T}} z^k d\mu \rightarrow 0$  as  $|k| \rightarrow \infty$ . Since  $\mu$  is finite by (a), it is a Rajchman measure.

(b) The set  $\{z^k\}$  is made up of unit vectors because  $\mu$  is a probability measure. Let  $\mu$  be singular with respect to the normalised Lebesgue measure  $\lambda$  on  $\mathfrak{R}_{\mathbb{T}}$ , with support  $[\mu] = C \subseteq \{z \in \mathbb{T} : z = e^{2\pi i \alpha} \text{ for every } \alpha \in [0, \frac{1}{4}]\}$ . So there exists an  $\mathfrak{R}_{\mathbb{T}}$ -measurable partition  $\{B, C\}$  of  $\mathbb{T}$  such that  $\mu(B) = \lambda(C) = 0$  and  $\lambda(\mathbb{T}) = \lambda(B) = \mu(C) = \mu(\mathbb{T}) = 1$ . Hence,

$$\langle z; 1 \rangle = \int_{\mathbb{T}} z d\mu(z) = \int_C z d\mu(z) = \int_{[0, \frac{1}{4}]} e^{2\pi i \alpha} d\eta(\alpha),$$

with  $\eta: \mathfrak{R}_{[0,1]} \rightarrow [0, 1]$  given by  $\eta(A) = \mu(\gamma(A))$  for every set  $A \in \mathfrak{R}_{[0,1]}$ , where  $\gamma: [0, 1] \rightarrow \mathbb{T}$  is the homeomorphism (thus a measurable transformation) of  $[0, 1]$  (equipped with the usual metric in  $\mathbb{R}$ ) onto  $\mathbb{T}$  (equipped with the arc-length metric in  $\mathbb{T}$ ) defined by  $\gamma(\alpha) = e^{2\pi i \alpha}$  for every  $\alpha \in [0, 1]$ . The support of  $\eta$  is  $[\eta] = \gamma^{-1}([\mu]) = \gamma^{-1}(C) \subseteq [0, \frac{1}{4}]$ . If  $\mu$  is continuous with respect to  $\lambda$ , then  $\eta$  is continuous with respect to the restriction to  $[0, 1]$  of the Lebesgue measure on  $\mathfrak{R}$ , and therefore singletons in  $[0, \frac{1}{4}]$  have  $\eta$ -measure zero. Therefore, since  $\eta$  is concentrated on  $[0, \frac{1}{4}]$ ,

$$\langle z; 1 \rangle = \int_{(0, \frac{1}{4})} \cos(2\pi \alpha) d\eta(\alpha) + i \int_{(0, \frac{1}{4})} \sin(2\pi \alpha) d\eta(\alpha).$$

As the functions  $\cos(2\pi \cdot)$  and  $\sin(2\pi \cdot)$  are strictly positive everywhere on  $(0, \frac{1}{4})$  and  $\eta((0, \frac{1}{4})) = \eta([0, \frac{1}{4}]) = \mu(C) = 1 > 0$ , the above integrals are strictly positive. So

$$\langle z; 1 \rangle \neq 0.$$

A similar argument shows that, for every  $0 \neq k \in \mathbb{Z}$ ,

$$\langle z^k; 1 \rangle = \int_{(0, \frac{1}{4})} e^{2k\pi i \alpha} d\eta(\alpha) \neq 0.$$

Consider again the position operator  $U_{\varphi, \mu}$  on  $L^2(\mathbb{T}, \mu)$ . Since  $U_{\varphi, \mu}$  is unitary, and so is  $U_{\varphi, \mu}^j$ , we get  $\langle z^j \psi; z^j \phi \rangle = \langle U_{\varphi, \mu}^j \psi; U_{\varphi, \mu}^j \phi \rangle = \langle \psi; \phi \rangle$  for every  $\psi, \phi$  in  $L^2(\mathbb{T}, \mu)$ , every  $j$  in  $\mathbb{Z}$ , and every measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$ . Thus, by the above inequality,

$$\langle z^{j+k}; z^j \rangle \neq 0 \text{ for every } j, k \in \mathbb{Z} \text{ with } k \neq 0.$$

(c) As we saw before, there are singular-continuous Rajchman measures on  $\mathfrak{R}_{\mathbb{T}}$ . Thus there are singular-continuous Rajchman probability measures concentrated on the first quadrant (compress and rotate its support if necessary). So (b) implies (c).  $\square$

**REMARK 6.2.** Just orthogonality is lost.

However, the standard application of the Stone–Weierstrass Theorem summarised in Remark 5.1 says that, for any finite measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$ , functions on  $L^p(\mathbb{T}, \mu)$  are approximated by the trigonometric polynomials in  $P(\mathbb{T})$ . In other words, using the

argument in the proof of Proposition 5.3 (for (a)  $\implies$  (b), and so for (a)  $\implies$  (e)), for every  $p \geq 1$ ,  $\bigvee_{k \in \mathbb{Z}} \{z^k\} = L^p(\mathbb{T}, \mu)$ , where  $\bigvee_{k \in \mathbb{Z}} \{z^k\}$  is the closure of the linear span of  $\{z^k\}$  (see, e.g., [5, Theorem 7.2] for the case of Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$ ).

## 7. RAJCHMAN MEASURES AND WEAK QUASISTABILITY

We say that a finite measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$  is *quasi-Rajchman* if

$$\liminf_{|k| \rightarrow \infty} \left| \int_{\mathbb{T}} z^k d\mu \right| = 0. \quad \text{Equivalently,} \quad \liminf_n \left| \int_{\mathbb{T}} z^n d\mu \right| = 0,$$

which means that there exists a subsequence  $\{n_j\}$  of the positive integers such that

$$\int_{\mathbb{T}} z^{n_j} d\mu \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Let  $\mu$  be a finite measure on  $\mathfrak{R}_{\mathbb{T}}$ . The Lebesgue Decomposition Theorem says that

$$\mu = \mu_a + \mu_s,$$

where  $\mu_a$  and  $\mu_s$  are the absolutely continuous and the singular parts of  $\mu$ , respectively. Moreover,  $\mu_s$  can be further decomposed as  $\mu_s = \mu_{sc} + \mu_{sd}$  yielding a refinement of the above decomposition (see, e.g., [11, Corollary 7.14]),

$$\mu = \mu_a + \mu_{sc} + \mu_{sd},$$

where  $\mu_{sc}$  and  $\mu_{sd}$  are the singular-continuous and singular-discrete (or simply discrete) parts of  $\mu$ . This is the *canonical decomposition* of  $\mu$ , where any of these parts may be absent. The continuous part  $\mu_c$  of  $\mu$  consists of the absolutely continuous and singular-continuous parts of  $\mu$ :

$$\mu_c = \mu_a + \mu_{sc}.$$

Recall that every Rajchman measure is continuous. Thus, if a measure on  $\mathfrak{R}_{\mathbb{T}}$  has a singular-discrete part, then it is not Rajchman. That is,

every singular-discrete measure on  $\mathfrak{R}_{\mathbb{T}}$  is not Rajchman.

In other words, for a singular-discrete measure  $\mu_{sd}$  on  $\mathfrak{R}_{\mathbb{T}}$ , we have  $\int_{\mathbb{T}} z^n d\mu_{sd} \not\rightarrow 0$  as  $n \rightarrow \infty$ . Then there exists at least one subsequence  $\{m_j\}$  of the positive integers for which  $\int_{\mathbb{T}} z^{m_j} d\mu_{sd} \not\rightarrow 0$  as  $j \rightarrow \infty$ . This is not the case for every subsequence. In fact,

singular-discrete measures on  $\mathfrak{R}_{\mathbb{T}}$  may be quasi-Rajchman or not.

The simplest examples are these. Set  $\mu_{sd} = \delta_1$ , the Dirac measure at  $1 \in \mathbb{T}$ , so that  $\int_{\mathbb{T}} z^n d\mu_{sd} = z^n|_1 = 1$  for all  $n \in \mathbb{N}$ , and hence there is no subsequence  $\{n_j\}$  for which  $\lim_j \int_{\mathbb{T}} z^{n_j} d\mu_{sd} = 0$ . On the other end, set  $\mu_{sd} = \delta_1 + \delta_{-1}$  so that  $\int_{\mathbb{T}} z^n d\mu_{sd} = z^n|_1 + z^n|_{-1} = 1 + (-1)^n$  for every  $n \in \mathbb{N}$ , and hence if  $\{n_j\}$  is the subsequence of all odd integers, then  $\int_{\mathbb{T}} z^{n_j} d\mu_{sd} = 0$  for every  $j \in \mathbb{N}$ .

Again, every Rajchman measure is continuous, but the converse fails. As we saw in Section 5, the Cantor–Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$  is singular-continuous but not Rajchman. A continuous measure, however, is always quasi-Rajchman.

**Lemma 7.1.** *If a finite measure on  $\mathfrak{R}_{\mathbb{T}}$  is continuous, then it is quasi-Rajchman.*

*Proof.* Let  $\mu$  be a finite measure on  $\mathfrak{R}_{\mathbb{T}}$ . The Wiener characterisation [21, Eq.(31) p.81] of continuous measures (see also, e.g., [15, p.365]) says that  $\mu$  is continuous if and only if the following mean of bounded summands goes to zero:

$$\mu \text{ is continuous} \iff \frac{1}{2k+1} \sum_{|\ell| \leq k} \left| \int_{\mathbb{T}} z^\ell d\mu \right| \rightarrow 0 \text{ as } |k| \rightarrow \infty,$$

that is,  $\frac{1}{2n+1} \left( \left| \int_{\mathbb{T}} d\mu \right| + \sum_{\ell=1}^n \left| \int_{\mathbb{T}} z^\ell d\mu \right| + \sum_{\ell=1}^n \left| \int_{\mathbb{T}} \bar{z}^\ell d\mu \right| \right) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,

$$\mu \text{ is continuous} \iff \frac{1}{2n+1} \sum_{\ell=1}^n \left| \int_{\mathbb{T}} z^\ell d\mu \right| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

since  $\left| \int_{\mathbb{T}} d\mu \right| = \int_{\mathbb{T}} d\mu = \mu(\mathbb{T})$ . If  $\mu$  is not quasi-Rajchman, then  $\liminf_n \left| \int_{\mathbb{T}} z^n d\mu \right| > 0$ , which means that there is an  $\varepsilon > 0$  and a positive integer  $\ell_\varepsilon$  such that  $\left| \int_{\mathbb{T}} z^\ell d\mu \right| > \varepsilon$  for every  $\ell > \ell_\varepsilon$ . Hence, for  $n > \ell_\varepsilon$ ,

$$\begin{aligned} \frac{1}{2n+1} \sum_{\ell=1}^n \left| \int_{\mathbb{T}} z^\ell d\mu \right| &= \frac{1}{2n+1} \left( \sum_{\ell=1}^{\ell_\varepsilon} \left| \int_{\mathbb{T}} z^\ell d\mu \right| + \sum_{\ell=\ell_\varepsilon+1}^n \left| \int_{\mathbb{T}} z^\ell d\mu \right| \right) \\ &> \frac{1}{2n+1} \left( \sum_{\ell=1}^{\ell_\varepsilon} \left| \int_{\mathbb{T}} z^\ell d\mu \right| \right) + \frac{n+\ell_\varepsilon}{2n+1} \varepsilon \geq \frac{n+\ell_\varepsilon}{2n+1} \varepsilon > \frac{1}{3} \varepsilon, \end{aligned}$$

so that  $\mu$  is not continuous. Thus if  $\mu$  is continuous, then it is quasi-Rajchman.  $\square$

The Cantor–Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$ , which is singular-continuous and was shown in Section 5 not to be Rajchman, is quasi-Rajchman by Lemma 7.1.

Since a measure is continuous if and only if it has no singular-discrete part in its canonical decomposition, an immediate consequence of Lemma 7.1 reads as follows.

**Corollary 7.2.** *If a finite measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$  has no singular-discrete part, then it is quasi-Rajchman.*

The converse fails: as we have seen before, there are singular-discrete finite measures  $\eta$  on  $\mathfrak{R}_{\mathbb{T}}$  for which  $\lim_j \int_{\mathbb{T}} z^{n_j} d\eta = 0$  for some subsequence  $\{n_j\}$ .

Proposition 5.3 shows that  $U_{\varphi, \mu}$  is weakly stable if and only if the measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$  is a Rajchman measure. Thus, if  $\mu$  is absolutely continuous, then  $U_{\varphi, \mu}$  is weakly stable. The next result establishes a counterpart for continuous measures.

**Theorem 7.3.** *If a finite measure  $\mu$  on  $\mathfrak{R}_{\mathbb{T}}$  is continuous, then the position operator  $U_{\varphi, \mu}$  on  $L^2(\mathbb{T}, \mu)$  is weakly quasistable.*

*Proof.* Let  $\mu$  be a finite continuous measure on  $\mathfrak{R}_{\mathbb{T}}$ , take a measurable nonnegative function  $g$  on  $\mathbb{T}$ , and consider the measure  $\eta_g$  on  $\mathfrak{R}_{\mathbb{T}}$  given by  $\eta_g(E) = \int_E g d\mu$  for  $E \in \mathfrak{R}_{\mathbb{T}}$ . Thus  $\eta_g$  is absolutely continuous with respect to  $\mu$ . So  $\eta_g$  is continuous with respect to Lebesgue measure on  $\mathfrak{R}_{\mathbb{T}}$  because  $\mu$  is. Also, suppose the nonnegative function  $g$  lies in  $L^1(\mathbb{T}, \mu)$ . Then  $\eta_g$  is finite. Hence, according to Lemma 7.1,

$$\eta_g \text{ is quasi-Rajchman.} \quad (*)$$

Moreover, for an arbitrary  $\mathbb{C}$ -valued measurable function  $\zeta$ , we get (cf. Remark 5.2)

$$\int_{\mathbb{T}} \zeta g d\mu = \int_{\mathbb{T}} \zeta d\eta_g. \quad (**)$$

By (\*), there is a subsequence  $\{n_j\}$  for which  $\lim_j \int_{\mathbb{T}} z^{n_j} d\eta_g = 0$ , and so by (\*\*)

$$\int_{\mathbb{T}} z^{n_j} g d\mu = \int_{\mathbb{T}} z^{n_j} d\eta_g \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Since this holds for an arbitrary nonnegative function  $g$  in  $L^1(\mathbb{T}, \mu)$ , take an arbitrary function  $\psi$  in  $L^2(\mathbb{T}, \mu)$  so that  $|\psi|^2$  is a nonnegative function in  $L^1(\mathbb{T}, \mu)$ . Then

$$\langle U_{\varphi,\mu}^{n_j} \psi; \psi \rangle = \langle z^{n_j} \psi; \psi \rangle = \int_{\mathbb{T}} z^{n_j} |\psi|^2 d\mu \rightarrow 0 \text{ as } j \rightarrow \infty \text{ for every } \psi \in L^2(\mathbb{T}, \mu).$$

Since  $L^2(\mathbb{T}, \mu)$  is a complex Hilbert space, Remark 3.5 ensures that

$$\langle U_{\varphi,\mu}^{n_j} \psi; \phi \rangle \rightarrow 0 \text{ as } j \rightarrow \infty \text{ for every } \psi, \phi \in L^2(\mathbb{T}, \mu). \quad \square$$

REMARK 7.4. A weakly quasistable operator with a common weak stability subsequence.

A weakly unstable operator that is weakly quasistable was exhibited in Example 4.1. Theorem 7.3 ensures the existence of a special class of such operators, namely,

weakly unstable unitary operators that are homogeneously weakly quasistable.

Indeed, for the position operator  $U_{\varphi,\mu}$  considered in the proof of Theorem 7.3, weak quasistability is achieved with a subsequence  $\{n_j\}$  that depends only on the fact that  $\mu$  is continuous according to Lemma 7.1. Also, the proof of Lemma 7.1 does not specify a subsequence  $\{n_j\}$  ensuring that  $\mu$  is quasi-Rajchman. Thus, that subsequence  $\{n_j\}$  of weak stability for  $U_{\varphi,\mu}$  in the proof of Theorem 7.3 does not depend on the function  $g$ , and so it does not depend on the function  $\psi$ . Then the same  $\{n_j\}$  is a subsequence of weak stability of all  $\psi$ . This holds in contrast to the general case of Remark 3.7(b). In other words, and according to Definition 3.3,

the position operator  $U_{\varphi,\mu}$  for a finite non-Rajchman continuous measure  $\mu$  (in particular, if  $\mu$  is not absolutely continuous) is unitary (so power bounded) and homogeneously weakly quasistable but not weakly stable.

We close the paper by giving a characterisation for non-Rajchman measures or, equivalently, for weakly unstable position operators.

**Corollary 7.5.** *Let  $\mu$  be a finite continuous measure on  $\mathfrak{R}_{\mathbb{T}}$ . If  $\mu$  is not Rajchman, then there exists a nontrivial subsequence  $\{m_j\}$  of the positive integers and a subsequence  $\{n_j\}$  of  $\{m_j\}$  such that*

$$\int_{\mathbb{T}} z^{m_j} d\mu \not\rightarrow 0 \text{ as } j \rightarrow \infty \quad \text{but} \quad \lim_j \int_{\mathbb{T}} z^{n_j} d\mu = 0.$$

*Proof.* If a finite measure on  $\mathfrak{R}_{\mathbb{T}}$  is not Rajchman, then  $\int_{\mathbb{T}} z^n d\mu \not\rightarrow 0$  as  $n \rightarrow \infty$ . So there is a subsequence  $\{m'_j\}$  of the positive integers for which

$$\int_{\mathbb{T}} z^{m'_j} d\mu \not\rightarrow 0 \text{ as } j \rightarrow \infty.$$

According to Lemma 7.1,  $\mu$  is quasi-Rajchman. So there is a subsequence  $\{n_j\}$  of the positive integers for which  $\{n_j\}$  and all its subsequences are such that

$$\int_{\mathbb{T}} z^{n_j} d\mu \rightarrow 0 \text{ as } j \rightarrow \infty.$$

(a) Now identify the subsequences  $\{m'_j\}$  and  $\{n_j\}$  of the positive integers with subsets of  $\mathbb{N}$ . Consider the subset  $\{m'_j\} \cup \{n_j\}$  of  $\mathbb{N}$  made up of all elements from  $\{m'_j\}$  and  $\{n_j\}$ . As a union of sets, the possibly common elements from  $\{m'_j\}$  and  $\{n_j\}$  appear only once in  $\{m'_j\} \cup \{n_j\}$ . Let this set be properly ordered so as to make it strictly increasing, and identify it with a subsequence of the positive integers, say  $\{m_j\}$ . Thus  $\{m_j\}$  is a supersequence of both  $\{m'_j\}$  and  $\{n_j\}$  and so

$$\int_{\mathbb{T}} z^{m_j} d\mu \not\rightarrow 0 \text{ as } j \rightarrow \infty.$$

(b) If  $\{m'_j\} \cup \{n_j\}$  is nontrivial, we are done. If not, we can make it nontrivial as follows. First recall that an arbitrary subsequence of the positive integers is nontrivial if and only if its complement in  $\mathbb{N}$  is another (infinite) subsequence. Thus the subsequence  $\{n_j\}$  is certainly nontrivial; otherwise,  $\mu$  would be Rajchman. Thus there exists a pair of complementary subsequences of  $\{n_j\}$ , say,  $\{n'_j\}$  and  $\{n''_j\}$ , both nontrivial with  $\{n'_j\} \cup \{n''_j\} = \{n_j\}$  and  $\{n'_j\} \cap \{n''_j\} = \emptyset$ , such that  $\int_{\mathbb{T}} z^{n'_j} d\mu \rightarrow 0$  and  $\int_{\mathbb{T}} z^{n''_j} d\mu \rightarrow 0$  as  $j \rightarrow \infty$ . Then  $\{m'_j\} \cup \{n'_j\}$  is nontrivial, and the argument in item (a) holds for  $\{n_j\}$  replaced with  $\{n'_j\}$ .  $\square$

Nontriviality for the sequence  $\{m_j\}$  and the fact that  $\{n_j\}$  is a subsequence of it are crucial in the above result, as well as in the next one. Corollary 7.5 can be rephrased in terms of the position operator  $U_{\varphi, \mu}$  on  $L^2(\mathbb{T}, \mu)$  according to Proposition 5.3(a,e) and Theorem 7.3, as follows.

**Corollary 7.6.** *Let  $\mu$  be a finite continuous measure on  $\mathfrak{R}_{\mathbb{T}}$ . If the position operator  $U_{\varphi, \mu}$  on  $L^2(\mathbb{T}, \mu)$  is weakly unstable, then there exists a nontrivial subsequence  $\{m_j\}$  of the positive integers and a subsequence  $\{n_j\}$  of  $\{m_j\}$  such that*

$$U_{\varphi, \mu}^{m_j} \psi \xrightarrow{w} 0 \text{ for some } \psi \in L^2(\mathbb{T}, \mu) \quad \text{but} \quad U_{\varphi, \mu}^{n_j} \psi \xrightarrow{w} 0 \text{ for every } \psi \in L^2(\mathbb{T}, \mu).$$

#### ACKNOWLEDGEMENT

I thank my co-authors in reference [9] for several discussions we had on Rajchman measures and weak stability.

#### REFERENCES

1. C. Badea and V. Müller, *On weak orbits of operators*, Topology Appl. **156** (2009), 1381–1385.
2. H. Bauer, *Measure and Integration Theory*, Walter de Gruyter, Berlin, 2001.
3. B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam, 1988.
4. A. Brown and C. Pearcy, *Introduction to Operator Theory I – Elements of Functional Analysis*, Springer, New York, 1977.
5. J.B. Conway, *Functions of One Complex Variable II*, Springer, New York, 1995.
6. S.R. Foguel, *A counterexample to a problem of Sz.-Nagy*, Proc. Amer. Math. Soc. **15** (1964), 788–790.
7. P.R. Halmos, *On Foguel's answer to Nagy's question*, Proc. Amer. Math. Soc. **15** (1964), 791–793.
8. P.R. Halmos, *A Hilbert Space Problem Book*, 2nd edn. Springer, New York, 1982.
9. Z.J. Jabłoński, I.B. Jung, C. Kubrusly, and J. Stochel, *Convergence of power sequences of operators via their stability*, Banach J. of Math. Anal. **18-4** (2024), paper 62, 27pp.
10. A.S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, Cambridge University Press, Cambridge, 1987.
11. C.S. Kubrusly, *Essentials of Measure Theory*, Springer-Switzerland, Cham, 2015.
12. C.S. Kubrusly, *Singular-continuous unitaries and weak dynamics*, Math. Proc. R. Ir. Acad. **116A** (2016), 45–56.
13. C.S. Kubrusly and B.P. Duggal, *Weak  $l$ -sequential supercyclicity and weak quasistability*, Rend. Circ. Mat. Palermo **73-2** (2024), 663–673.
14. C.S. Kubrusly and P.C.M. Vieira, *Weak stability and quasistability*, Rend. Circ. Mat. Palermo **73-8** (2024), 3217–3228.
15. R. Lyons, *Seventy years of Rajchman measures*, J. Fourier Anal. Appl. (Kahane Special Issue) (1995), 363–377.
16. R. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 1998.
17. M.D. Menchoff, *Sur l'unicité du développement trigonométrique*, CR Acad. Sc. Paris **163**(2) (1916), 433–436.

18. L. Neder, *Über die Fourierkoeffizienten der Funktionen von beschränkter Schwankung*, Math. Z. **6** (1920), 270–273.
19. A. Rajchman, *Une classe de séries trigonométriques qui convergent presque partout vers zéro*, Math. Ann. 101, (1929), 686–700.
20. F. Riesz *Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung*, Math. Z. **2** (1918), 312–315.
21. N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, Journal of Mathematics and Physics **3**(2) (1924), 72–94.

CATHOLIC UNIVERSITY OF RIO DE JANEIRO, BRASIL  
*E-mail address:* carlos@ele.puc-rio.br